矩陣的乘法結合律的證明

（資料來源：龍騰教師手冊）

若 A，B，C 為矩陣，且 $(AB)C$ 與 $A(BC)$ 都有意義，則 $(AB)C=A(BC)$．

證　設 $A=\left[a_{ij}\right]_{m\times n}$，$B=\left[b_{ij}\right]_{n\times p}$，$C=\left[c_{ij}\right]_{p\times q}$．

由矩陣的乘法定義，得

$$AB=\left[\sum_{k=1}^{n}a_{ik}b_{kj}\right]_{m\times p}，\quad (AB)C=\left[\sum_{j=1}^{p}\left(\sum_{k=1}^{n}a_{ik}b_{kj}\right)c_{jt}\right]_{m\times q}，$$

$$BC=\left[\sum_{j=1}^{p}b_{kj}c_{jt}\right]_{n\times q}，\quad A(BC)=\left[\sum_{k=1}^{n}a_{ik}\left(\sum_{j=1}^{p}b_{kj}c_{jt}\right)\right]_{m\times q}．$$

由實數的加法結合律、乘法結合律與分配律，得

$$\sum_{k=1}^{n}a_{ik}\left(\sum_{j=1}^{p}b_{kj}c_{jt}\right)$$

$$=a_{i1}\left(\sum_{j=1}^{p}b_{1j}c_{jt}\right)+a_{i2}\left(\sum_{j=1}^{p}b_{2j}c_{jt}\right)+\cdots+a_{in}\left(\sum_{j=1}^{p}b_{nj}c_{jt}\right)$$

$$=a_{i1}\left(b_{11}c_{1t}+\cdots+b_{1p}c_{pt}\right)+a_{i2}\left(b_{21}c_{1t}+\cdots+b_{2p}c_{pt}\right)+\cdots+a_{in}\left(b_{n1}c_{1t}+\cdots+b_{np}c_{pt}\right)$$

$$=\left(a_{i1}b_{11}c_{1t}+\cdots+a_{i1}b_{1p}c_{pt}\right)+\left(a_{i2}b_{21}c_{1t}+\cdots+a_{i2}b_{2p}c_{pt}\right)+\cdots+\left(a_{in}b_{n1}c_{1t}+\cdots+a_{in}b_{np}c_{pt}\right)$$

$$=\left(a_{i1}b_{11}c_{1t}+\cdots+a_{in}b_{n1}c_{1t}\right)+\left(a_{i1}b_{12}c_{2t}+\cdots+a_{in}b_{n2}c_{2t}\right)+\cdots+\left(a_{i1}b_{1p}c_{pt}+\cdots+a_{in}b_{np}c_{pt}\right)$$

$$=\left(\sum_{k=1}^{n}a_{ik}b_{k1}\right)c_{1t}+\left(\sum_{k=1}^{n}a_{ik}b_{k2}\right)c_{2t}+\cdots+\left(\sum_{k=1}^{n}a_{ik}b_{kp}\right)c_{pt}$$

$$=\sum_{j=1}^{p}\left(\sum_{k=1}^{n}a_{ik}b_{kj}\right)c_{jt}\ ．$$

最後一式為 $(AB)C$ 的第 (i,j) 元，所以得知 $(AB)C=A(BC)$．